N71-30854

NASA TECHNICAL TRANSLATION NASA TT F-13,720

THE SAMPLING THEOREM FOR THE IMAGE OBTAINED BY A CIRCULAR APERTURE AND ITS APPLICATIONS TO NUMERICAL CALCULATION OF AMPLITUDE, INTENSITY AND THEIR FOURIER TRANSFORMS WITH ANALYTICAL EXPRESSION OF RESPONSE FUNCTIONS

H. Gamo

Translation of: "Enkei Kaikō ni yoru Zō ni tai-suru Sanpuringn Teiri to sono Sūka Keisan e no Ōyō," Journ. of Appl. Phys. of Japan (Oyobutsuri), Vol. 26, 1957a, pp. 102-115.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D. C. 20546 JULY 1971

THE SAMPLING THEOREM FOR THE IMAGE OBTAINED BY A CIRCULAR APERTURE AND ITS APPLICATIONS TO NUMERICAL CALCULATION OF AMPLITUDE, INTENSITY AND THEIR FOURIER TRANSFORMS WITH ANALYTICAL EXPRESSION OF RESPONSE FUNCTIONS[1]

Hideya Gamo[2]

ABSTRACT. The sampling theorem for the amplitude of waves by a circular aperture is derived; namely, the complex amplitude F(ρ, ϕ) in the image plane is expressed as /102*

$$F(\rho,\varphi)=\sum_{n=-\infty}^{+\infty}\sum_{s=1}^{+\infty}F_n(\lambda_{ns}/k\alpha)C_{ns}(\rho,\varphi) \tag{1}$$

$$F_n(\lambda_{ns}/k\alpha)=\frac{1}{2\pi}\int_0^{2\pi}F(\lambda_{ns}/k\alpha,\varphi)\exp(-in\varphi)\,d\varphi \tag{2}$$

$$C_{ns}(\rho,\varphi)=\exp(in\varphi)J_n(k\alpha\rho)2\lambda_{ns}/J_n'(\lambda_{ns})\{(k\alpha\rho)^2-\lambda_{ns}^2\} \tag{3}$$

where α is aperture constant, ρ,ϕ polar coordinates, $k = 2\pi/\lambda$ and λ wavelength, λ_{ns} s^{th} zero of the Bessel function $J_n(x)$. The sampling functions (C_{ns}) satisfy the orthogonal relation

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}C_{ns}(\rho,\varphi)C_{mt}^*(\rho,\varphi)\rho\,d\rho\,d\varphi=2\delta_{nm}\delta_{st}/\{k\alpha J_n'(\lambda_{ns})\}^2 \tag{4}$$

and C_{ns} is unity upon a sampling circle of radius $\lambda_{ns}/k\alpha$ and is zero upon the other sampling circles of the same order (Figure 1). The sampling coefficient $F_n(\lambda_{ns}/k\alpha)$ is obtained by the integration with angle ϕ of the complex amplitude at a sampling circle multiplied by exp(-inϕ). At each sampling circle of order zero a sampling coefficient is obtained, and at each sampling circle of

*Numbers in the margin indicate pagination in the original foreign text.

(1) Part of the contents was published in "Kagaku", Dec., 1956.

(2) Department of Physics, University of Tokyo.

non-zero order two sampling coefficients $F_n(\lambda_{ns}/k\alpha)$ and $F_{-n}(\lambda_{ns}/k\alpha)$ are obtained. There is an important relation between the above sampling coefficient and the coefficient of Fourier-Bessel expansion of the pupil function;

$$K_{ns} = 4\pi F_n(\lambda_{ns}/k\alpha)/(i)^n\{k\alpha J_n'(\lambda_{ns})\}^2 \tag{5}$$

and the pupil function $f(r, \theta)$ is expressed as

$$f(r,\theta) = \sum_{n=-\infty}^{+\infty}\sum_{s=1}^{+\infty} K_{ns}\exp(in\theta)J_n(\lambda_{ns}r/\alpha) \tag{6}$$

The number of sampling coefficients whose sampling circles are included within a circle of area S of the image, namely, the number of degrees of freedom, is estimated as $\pi\alpha^2 S/\lambda^2$ by considering the distribution of zeros of $J_n(x)$ and by using known results for rectangular apertures.

That the Fourier transform of intensity distribution of an image by a circular aperture α vanishes outside the region of a circle of radius 2α in various degrees of coherence of illumination, is shown by considering that the intensity is described by a series of products $C_{ns}C_{mt^*}$, and Fourier transforms of $C_{ns}C_{mt^*}$ vanish outside the circle mentioned above, as is clarified by means of convolution integrals or analytically in Appendix 2. Because of the limited spectrum of the intensity distribution mentioned above, the sampling theorem for intensity distribution is obtained by putting 2α in place of α into Equations (1) - (3). The Fourier transform of intensity distribution is obtained by (5) and (6) where α is replaced by 2α, and the response function may be calculated by these equations from the intensity distribution obtained analytically or experimentally.

These sampling theorems may be used for interpolation of amplitudes and intensities. By taking the well known Airy figure at sharp focus as an example, it is shown that the sampling theorem for intensity will be preferred to the sampling for amplitude because of the higher accuracy of the former. The relation between the circle polynomial expansion due to Zernike, Nijboer and Hienhuis and the Fourier-Bessel expansion of the pupil function is considered as a preliminary to numerical calculation by sampling theorems.

The response function of a pupil with small aberrations may be expressed analytically by the Fourier transform of analytical expression for intensity distribution, and infinite response functions obtained in this paper, namely, the method using (5) and (6), the one using Fourier transforms of $C_{ns}C_{mt*}$ in Appendix 2, and the one treated in Appendix 3 are discussed. The first method will be most conveniently used and be complemented by the third method, since the value of the response function at the origin which cannot be obtained accurately by the first method is easily obtained by the third method.

1. INTRODUCTION

Most of the optical images we treat are two-dimensional; the great majority of them can be obtained by a circular aperture. Among the two-dimensional theorems, there are those which are treated by Blanc-Lapieree, Gabor, Toraldo di Francia, Fellgetl and Linfoor [2]. They correspond to a square aperture; the amplitude at each lattice point of the two-dimensional square lattice is taken as the sampling value. It is of course possible to treat the images by a circular aperture with this method. For example, the sampling values corresponding to the square aperture which circumscribes the circular aperture under consideration need to be taken. It has faults, such as the fact that the sampling values are not completely independent, and also, when there is axial symmetry to the image, it is difficult to see through. Hence, it is imperative that a sampling theorem which best fits the circular aperture system be derived. Since sampling theorems generally have meanings in the interpolation method, the derivation of a sampling theorem for an axial symmetric circular aperture can be employed as an interpolation method for calculation of the analytical images of the Zernike, Nijboer, and Nienhuis' [3] system with aberrations. The sampling theorem for the intensity distribution discussed in 2.2 can be actually utilized. By constructing a table of the standard sampling functions (C_{ns} in the text), any intensity distribution for the given sampling value can be calculated by multiplication and addition.

Since the Fourier transform of the standard functions can be closely correlated with the Fourier-Bessel expansion, the Fourier transform can be obtained by giving the sampling values. By applying this method to the calculation of Zernike, the response function can be obtained. It probably can be said that this method is an alternative for the convolution integral method by Hopkins, De [4]. Also, if the analytical image of the light source can be obtained by experiments, the response function can be derived by determining the sampling value of the image. The sampling theorem for intensity is, also, useful for arranging experimental data.

We will consider the degrees of freedom of the circular aperture image. This problem is related to the distribution of zeros of the Bessel function. Although the calculation of Bessel functions becomes necessary at several places, references are given for the basic equations [5]. In order to prove that the frequency band of the intensity distribution is restricted, aside from the direct observation method, the series expansion method is given in the appendix. They were added since they cannot be found in reference books on Bessel functions, and also it was thought that they might be useful in the future. It is hoped that the physical meanings of the results are clear without getting involved in discussions of the equations.

2. SAMPLING THEOREMS

2.1 Sampling Theorems of the Amplitude(3)

Letting α be the aperture constant in optics, only those Fourier components included in a circle of radius α contribute to the image formation. $\alpha = \sin\theta$, where θ is half of the angle of the light flux entering the pupil from a point on the object surface. The Fourier component restricted in the

(3) According to the private communication with D. Gabor, the sampling theorem for a circular aperture was given in the Ritchie Lecture (1952) which, however, is unpublished.

circle of radius α is expressed as $f(r,\theta)$. r,θ are the radius and angle, respectively, in polar coordinates. $f(r,\theta)$ is always zero for $r \geq \alpha$ and takes values other than zero for $r < \alpha$. In contrast to the use of the double /104 Fourier series for deriving the sampling theorem for a square aperture, the Fourier-Bessel expansion in the circle of a circular aperture $f(r,\theta)$ is used.

The Fourier-Bessel function is expressed by the following equation:

$$f(r,\theta)=\sum_{n=-\infty}^{+\infty}\sum_{s=1}^{+\infty}K_{ns}\exp(in\theta)J_n(\lambda_{ns}r/\alpha) \tag{1}$$

The expansion coefficient K_{ns} is given by the following orthogonal relation.

$$K_{ns}=\frac{1}{\pi\alpha^2\{J_n'(\lambda_{ns})\}^2}\int_0^{2\pi}\int_0^{\alpha}f(r,\theta)\times\exp(-in\theta)J_n(\lambda_{ns}r/\alpha)\,r\,dr\,d\theta \tag{2}$$

λ_{ns} is the s^{th} zero point in the Bessel function of the first kind $J_n(x)$. The orthogonal relation is obtained by the Lommel integral (Appendix 1).

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\alpha}\exp[i(n-m)\theta]J_n(\lambda_{ns}r/\alpha)\times J_m(\lambda_{mt}r/\alpha)\,r\,dr\,d\theta=\delta_{nm}\delta_{st}\{\alpha J_n'(\lambda_{ns})\}^2/2 \tag{3}$$

Let us obtain the Fourier transform of the pupil function $f(r,\theta)$ given by the above Fourier-Bessel expansion. Needless to say, the Fourier transform gives the complex amplitude of the waves on the image. That is, the complex amplitude $F(\rho,\phi)$ of the image is

$$F(\rho,\varphi)=\left(\frac{k}{2\pi}\right)^2\int_0^{2\pi}\int_0^{\alpha}f(r,\theta)\times\exp[ik\rho r\cos(\theta-\varphi)]\,r\,dr\,d\theta \tag{4}$$

where $k = 2\pi/\lambda$; λ is the wavelength. Substituting (1) into (4) and integrating by terms,

$$F(\rho,\varphi)=\sum\sum K_{ns}\alpha^2(i)^n\frac{\lambda_{ns}J_n'(\lambda_{ns})}{(k\alpha\rho)^2-\lambda_{ns}^2}\times J_n(k\alpha\rho)\exp[in\varphi] \tag{5}$$

(cf. Appendix 1). In the following (5) is rearranged in the sampling theorem form. Let us first define the following function $C_{xs}(\rho,\phi)$ as a standard function for the expansion of an image

$$C_{ns}(\rho,\varphi)=\frac{2\lambda_{ns}}{J_n'(\lambda_{ns})}\frac{J_n(k\alpha\rho)}{(k\alpha\rho)^2-\lambda_{ns}^2}\exp(in\varphi) \tag{6}$$

The function becomes 1 on the circle of radius $\rho=\lambda_{ns}/k\alpha$, and zero on the circle of radius $\rho=\lambda_{nt}/k\alpha\ (t\neq s)$ belonging to the same order n. (see Figure 1). Among the functions $\{C_{ns}(\rho,\phi)\}$, the following orthogonal relation holds,

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}C_{ns}(\rho,\varphi)C_{mt}^*(\rho,\varphi)\rho d\rho d\varphi = 2\delta_{nm}\delta_{st}/\{k\alpha J_n'(\lambda_{ns})\}^2 \tag{7}$$

(See the end of Appendix 1, 2 for the proof.) The properties of the function $\{C_{ns}(\rho,\phi)\}$ are common, to a certain extent, to those of the standard function $\sin(kax - n\pi)\sin(kay - m\pi)/(kaz - n\pi)(kay - m\pi)$ of a square aperture.

The amplitude $F(\rho,\phi)$ of an image is expanded by the above orthogonal function $\{C_{ns}(\rho,\phi)\}$, and becomes

$$F(\rho,\varphi)=\sum_{n=-\infty}^{+\infty}\sum_{s=1}^{+\infty}F_n(\lambda_{ns}/k\alpha)C_{ns}(\rho,\varphi) \tag{8}$$ [4]

The expanded coefficient $F_n(\lambda_{ns}/k\alpha)$ is given by:

$$F_n(\lambda_{ns}/k\alpha)=\frac{1}{2\pi}\int_0^{2\pi}F(\lambda_{ns}/k\alpha,\varphi)\exp(-in\varphi)d\varphi \tag{9}$$

<u>This is the sampling theorem for the image by a circular aperture</u>. The expansion coefficient $F_n(\lambda_{ns}/k\alpha)$ has a magnitude equal to that obtained by multiplying $\zeta\exp(-in\phi)$ to the amplitude of $F(\lambda_{ns}/k\alpha,\phi)$ on the circle (sampling circle of radius $\lambda_{ns}/k\alpha$, and integrating all around with respect to the angle ϕ. These are our sampling values.

---

[4] For a special case when n = 0, that is when it is axially symmetric, Kokura of Koana Lab. independently derived the same result (unpublished).

The sampling values can be defined by the integral form with respect to cos $m\phi$, sin $m\phi$ [see (40)]. Equation (8) can be derived from the conventional definition of the expansion coefficient[5].

$$F_n(\lambda_{ns}/k\alpha)=\frac{\{k\alpha J_n'(\lambda_{ns})\}^2}{2}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}F(\rho,\varphi)\times C_{ns}^*(\rho,\varphi)\rho d\rho d\varphi \tag{10}$$

As evident by comparing the sampling theorem (7) and Equation (5) obtained by the Fourier-Bessel expansion, the following relation exists between the coefficient K_{ns} of the Fourier-Bessel expansion and the sampling value $F_n(\lambda_{ns}/k\alpha)$ of the sampling theorem:

$$K_{ns}=4\pi F_n(\lambda_{ns}/k\alpha)/(i)^n\{k\alpha J_n'(\lambda_{ns})\}^2 \tag{11}$$

This is an important equation which is used to obtain the amplitude of the Fourier transform — that is, the pupil function $f(r,\theta)$ from the given sampling value $F_n(\lambda_{ns}/k\alpha)$. The pupil function $f(r,\theta)$ is expressed by the sampling value of the image $F_n(\lambda_{ns}/k\alpha)$ as follows,

$$f(r,\theta)=\sum_{n=-\infty}^{+\infty}\sum_{s=1}^{+\infty}4\pi F_n(\lambda_{ns}/k\alpha)\times\exp(in\theta)J_n(\lambda_{ns}r/\alpha)/(i)^n\{k\alpha J_n'(\lambda_{ns})\}^2 \tag{12}$$

Since the intensity distribution of the image is equal to the square of the absolute value of the amplitude, we have

$$I(\rho,\varphi)=\sum_{n,s}\sum_{m,t}F_n(\lambda_{ns}/k\alpha)\times F_m^*(\lambda_{mt}/k\alpha)C_{ns}(\rho,\varphi)C_{mt}^*(\rho,\varphi) \tag{13}$$

The integrated quantity I_0 of the intensity for the entire image by the orthogonality (9) is,

$$I_0=\sum_{n=-\infty}^{+\infty}\sum_{s=1}^{+\infty}4\pi|F_n(\lambda_{ns}/k\alpha)|^2/\{k\alpha J_n'(\lambda_{ns})\}^2 \tag{14}$$

---

(5) For example, by expressing $F_n(\lambda_{ns}/k\alpha)$ with $f(r,\theta)$ from (11) and (2), and, on the other hand, by assuming $F(g_{ns}/k\alpha\ \phi)$ of (8) can be converted to the Fourier transform in $f(r,\theta)$ and substituting, yields the equation agreeing with the former equation.

Although the integral intensity I_0 is equal to the square of the absolute value of each sampling value according to the sampling theorem for a square aperture, according to that for a circular aperture it is equal to the square of the absolute sampling value multiplied by the coefficient /105
$4\pi/k\{J_n'(\lambda_{ns})\}^2$.

When the sampling value $\{F_n(\lambda_{ns}/k\alpha)\}$ is given, and when an amplitude distribution equal to $F_n(\lambda_{ns}/k\alpha)$ is given on a circle of radius $\exp(in\theta)$ $4\pi/\lambda_{ns}k\alpha\{J_n'(\lambda_{ns})\}^2$ on an object, the amplitude distribution of an image obtained by the initial circular aperture agrees with the amplitude distribution of the image before performing the sampling. Such a property is common to the case of sampling for a square aperture. The only difference is that it is multiplied by the coefficient $4\pi/\lambda_{ns}k\alpha\{J_n'(\lambda_{ns})\}^2$.

As an example of the above theorem, let us consider the cases when the amplitude distribution of the image $F(\rho,\phi) = 1$ and $\exp(in\theta)$. For each case employing (8) and (9), the following partial fraction expansions of $J_0(z)$, $J_n(z)$ are obtained

$$J_0(z) = 2\sum_{s=1}^{+\infty} \lambda_s/J_0'(\lambda_s)(z^2-\lambda_s^2)$$

$$J_n(z) = 2\sum_{s=1}^{+\infty} \lambda_{ns}/J_n'(\lambda_{ns})(z^2-\lambda_{ns}^2)$$

These equations, of course, can be proved directly.

2.2 Degrees of Freedom of Images by Circular Aperture

The number of sampling values included in a domain (area S) on an image is the "number of degrees of freedom" of the domain. Let us consider a domain as a circle of radius a, and examine how many circles of radius $\lambda_{ns}/k\alpha$ can be included in this circle. When $n = 0$, there is one sampling value on the circle, whereas for $n \neq 0$, there are two sampling values $F_n(\lambda_{ns}/k\alpha)$ and $F_{-n}(\lambda_{ns}/k\alpha)$. The problem can be solved by examining the zero distribution of the Bessel function $J_n(x)$ ($n = 0, 1, 2, \ldots$).

Although an explicit description regarding zeros can be found in Chapter 15 of Watson's book, or in the table by Jahnke-Emde, a general expression cannot be found which holds for all n and s. Hence, at the present, it is not possible to determine the exact number of degrees of freedom from the distribution of zeros. Let us, therefore, utilize the results obtained from the asymptotic expansion of $J_n(x)$ can be expressed by the following asymptotic equation

$$J_n(x) \longrightarrow \sqrt{\frac{1}{\pi x}} \cos\left(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right) \tag{15}$$

Hence, the positive zero λ_{ns} is,

$$\lambda_{ns} \fallingdotseq \left(\tfrac{1}{2}n + s - \tfrac{1}{4}\right)\pi \tag{16}$$

(Gray, Mathew, p. 86). Based on the exact values of zeros, $\lambda_{ns}/\pi - 1/2\ n$ and $1/2\ n$ are plotted on x and y coordinates, respectively. It is convenient to learn the trend of zero distribution. For a given degree n, the approximation of (16) holds for zeros of high order s. Although the deviation from (16) becomes larger for the low order zeros, the validity of (16) is preserved. In Figure 2, since the values of zeros are expressed by $\lambda_{ns} = \pi(x+y)$, the total number of zeros of $\lambda_{ns} < \alpha$ is equal to the number of points contained between axes X,Y and the line intersecting the axes at 45° at $(\alpha/\pi, 0), (0, \alpha/\pi)$. For simplicity, let us assume (20) (Figure 2) and examine the total number of zeros smaller than $\alpha = k\alpha\rho_0$. Let N be the positive integers which satisfy /106

$$N > k\alpha\rho_0/\pi > N-1$$

As is evident from the figure, the total number of zeros contained between the axes and the line intersecting axes X,Y respectively at (N,0)(0,N) at 45° is

$$\sum_{r=1}^{N} r + \sum_{r=1}^{N-1} r = N^2$$

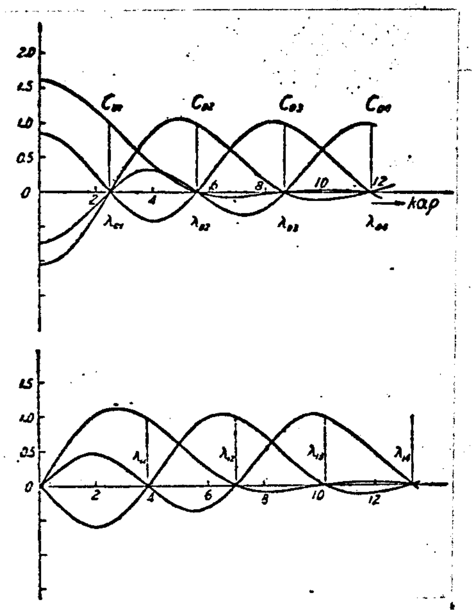

Fig. 1 Showing the sampling functions $C_{0s}(\rho,\varphi)$ and $C_{1s}(\rho,\varphi)$ where λ_{0s} and λ_{1s} are S-th zero of Bessel function $J_0(x)$ and $J_1(x)$ respectively.

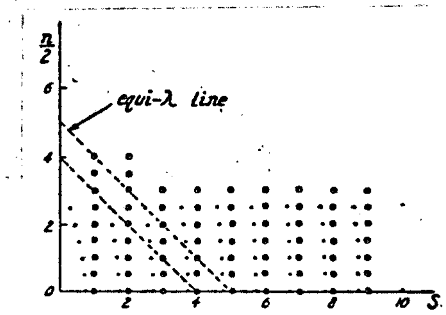

Fig. 2 Showing the distribution of zeros λ_{ns} of Bessel function $J_n(x)$ by blak points whose coordinates are given by $x=\lambda_{ns}/\pi-n/2$ and $y=n/2$, and the distribution of approximated zeros of $\pi(n/2+s)$ by circles.

Let us consider now the number of sampling values at each sampling circle and obtain the total number of sampling values F contained in a circle of radius $\alpha = k\alpha\rho_0$.

$$F=\sum_{r=0}^{N-1}(1+2r)+\sum_{r=0}^{N-2}(1+2r)=N^2+(N-1)^2$$

Let $F = 2N^2$ and $N = k\alpha\rho_0/\pi$, then $F = 2(k\alpha\rho)^2/\pi^2$. Substituting $k = 2\pi/\lambda$ and $\pi\rho_0^{\,2} = S$, we have

$$F=8\alpha^2S/\pi\lambda^2 \tag{17}$$

Comparing the exact λ_{ns} of Figure 2 and the distribution of the approximate values, the right number of the degrees of freedom is slightly larger than the approximate values from (17).

Let us now consider the degrees of freedom from a different angle. As is well known, the degrees of freedom of a square aperture is $F = 4\alpha^2S/\lambda^2$, where 2α is the side of the aperture. Comparing the degrees of freedom of a square aperture which circumscribes and one that inscribes a circle, the desired degrees of freedom F must be in between, that is,

$$4a^2S/\lambda^2 > F > 2a^2S/\lambda^2 \tag{18}$$

The approximation (17) also satisfies the relation in (18). We will now consider a square which is inscribed in the above circle, and the rectangular apertures which fit between the circle and the square. The additional degrees of freedom arising from such a treatment will be added to that from the inscribed aperture $2n^2S/\lambda^2$. At the limit of approximation, we obtain.

$$F = \pi a^2 S/\lambda^2 \tag{19}$$

This is equivalent to determining the area of a circle whose radius is α. The ratio between F from (19) and the approximation from (17) is $\pi^2/8 = 1.233$. If one were to start with a more accurate distribution of zeros, values closer to those from (19) would probably be obtained.

2.3. Sampling Theorem for Intensities

Since intensity is what we directly observe, the sampling theorem for intensity distribution is more practical than that for the amplitude. The intensity distribution of images is given by (13) when illuminated by coherent light. When the illumination is incoherent or semicoherent, the intensity distribution of an image is equal to substituting the mean values at each point instead of the coefficient of (13) $F_n(\lambda_{ns}/k\alpha)F_{m*}(\lambda_{mt}/k\alpha)$. We call the matrix composed of these coefficients as "intensity matrix"[6]. The frequency band of the intensity distribution $I(\rho,\phi)$ is determined by the bandwidth of $\{C_{ns}(\rho,\phi)C_{mt*}(\rho,\phi)\}$, and not by the interference of light.

Let us consider the bandwidth of $C_{ns}(\rho,\phi)C_{mt*}(\rho,\phi)\}$. Let us express C_{ns}, C_{mt*} in rectangular coordinates. Let each be expressed by $F(x,y)G(x,y)$,

---

[6] See the previous paper [6] and the paper which is being submitted.

and the Fourier transforms be f(u,v), g(x,y). Then the Fourier transform of the product F(x,y)G(x,y) is given by the convolution (Faltung) integral between f(u,v)g(u,v)

$$D(u_0, v_0) = \frac{1}{(2\pi)^2} \iint g(u,v) f(u_0 - u, v_0 - v)\, du\, dv \tag{20}$$

according to Parseval's theorem (Titchmarsh [7], p. 51), where $f(u_0 - u, u_0 - v)$, $g(u,v)$ assumes the non-zero values within circles of radius α whose centers are at (u_0, v_0) $(0,0)$, respectively. It is always zero at the outside. Hence, the above integral $D(u_0, v_0)$ is completely zero at $\sqrt{u_0^2 + v_0^2} \geq 2\alpha$. This indicates that the frequency band of the product F(x,y) G(x,y) — that is, of the product $C_{ns}(\rho,\phi)C_{mt*}(\rho,\phi)$ — is 2α.

<u>The frequency band of the intensity distribution of the image given by the aperture α is restricted within the circle of radius 2α regardless of the illumination coherency.</u>

As indicated in Appendix 2, the Fourier transform of the product $C_{ns}C_{mt*}$ can be obtained as a series of Bessel functions from a different angle than the above proof[(7)].

Since we now know that the frequency band of the intensity distribution is restricted within a circle whose radius is twice the aperture α, replacing $\alpha \to 2\alpha$ $F \to I(\rho,\phi)$ in the sampling theorems (7), (8) for the amplitude in Appendix 2, the sampling theorem for the intensity can be directly derived. Furthermore, since the intensity $I(\rho,\phi)$ is always a positive integer, regardless of ρ,ϕ, the relationship

$$F_{-n}(\lambda_{ns}/2k\alpha) = F_n^*(\lambda_{ns}/2k\alpha) = A_{ns} - iB_{ns}$$

[(7)] By expressing Equation (20) in polar coordinates, writing the pupil function in Fourier-Bessel expansion and integrating by terms, the series of the response function can be obtained which has the sampling values of the amplitude as coefficients. Here, the addition theorem of the Bessel function and Lommel integral will be used. It will be supplemented during the proof-reading on Feb. 27.

exists among the sampling values. It follows that,

$$I(\rho,\varphi)=\sum_{s=1}^{+\infty}A_{0s}P_{0s}(\rho)$$
$$+2\sum_{n}\sum_{s=1}^{+\infty}(A_{ns}\cos n\varphi-B_{ns}\sin n\varphi)P_{ns}(\rho) \tag{21}$$

where,

$$\left.\begin{aligned}A_{ns}&=\frac{1}{2\pi}\int_0^{2\pi}I(\lambda_{ns}/2k\alpha,\varphi)\cos n\varphi\,d\varphi\\ B_{ns}&=\frac{1}{2\pi}\int_0^{\pi}I(\lambda_{ns}/2k\alpha,\varphi)\sin n\varphi\,d\varphi\end{aligned}\right\} \tag{22}$$

$$P_{ns}(\rho)=\frac{2\lambda_{ns}}{J_n'(\lambda_{ns})}\frac{J_n(2k\alpha\rho)}{(2k\alpha\rho)^2-\lambda_{ns}^2} \tag{23}$$ /107

Since the coefficients of Fourier-Bessel expansion are given by (11), the Fourier transform $i(r,\theta)$ of the intensity distribution can be obtained.

$$i(r,\theta)=4\pi\sum_{s=1}^{+\infty}A_{0s}J_0(\lambda_{0s}r/2\alpha)/\{k\alpha J_0'(\lambda_{0s})\}^2$$
$$+8\pi\sum_{n}\sum_{s=1}^{+\infty}(A_{ns}\cos n\theta-B_{ns}\sin n\theta)$$
$$\times J_n(\lambda_{ns}r/2\alpha)/(i)^n\{k\alpha J_n'(\lambda_{ns})\}^2 \tag{24}$$

Similarly to the previous section, the sampling values of the intensity contained in a circle of area S is

$$F(\text{intensity})=4\pi\alpha^2S/\lambda^2 \tag{25}$$

Here, the bandwidth of the intensity is given by the radius 2α. Regardless of the coherency of illumination, the sampling values within the area S are given by (25).

Nevertheless, the above sampling values A_{ns} and B_{ns} cannot assume any arbitrary value independently, unlike the case of amplitude; the intensity $I(\rho,\phi)$ given by (21) must always be positive regardless of the value of ρ,ϕ. There is a certain degree of redundancy among the sampling values of the intensity distribution. Since F (intensity) of Equation (25) does not give independent sampling values, its meaning is restricted compared to F values for the amplitude distribution in the previous section. The redundancy in the sampling values of the intensity distribution is an advantage in numerical calculations. This will become clear in the following section.

3. APPLICATION OF THE SAMPLING THEOREM TO NUMERICAL CALCULATIONS

3.1 Airy Figure

The above sampling theorem is a linear interpolation method from the standpoint of numerical calculations. It is the case where the necessary sampling value is a minimum. Once the table of $\{C_{ns}(\rho,\phi)\}$ is given, the rest is multiplication and addition of $C_{ns}(\rho,\phi)$ for the sampling values $F_n(\lambda_{ns}/k\alpha)$. Hence, in the "light contour" numerical calculation [3] of Zernike, Nijboer and Nienhuis, which is discussed in the next section, only the sampling value $F_n(\lambda_{ns}/k\alpha)$ need be determined, while the rest of the values can be obtained by the above interpolation method.

In order to expect rigorous agreement of the calculation results with the amplitude before sampling, an infinite number of sampling values is required, as is evident from (8). In the actual case, the sampling number must be made finite. Let us examine the degree of approximation with a finite number of sampling values. The one which satisfies the objective is probably diffraction by a circular aperture (Airy Figure) [8] for which rigorously calculated values are known, and which is free of aberrations and off-focusing. In the Airy Figure calculations, only the terms which contain $F_0(\lambda_{os}/k\alpha)$, $C_{os}(\rho,\phi)$ $(s = 1, 2, \ldots)$ in $r \leqq \alpha$ need be considered, since the system has axial symmetry. In this case, the pupil function $f(r,\theta)$ is $f(r,\theta) = 1$ for $r \leqq \alpha$, and $f(r,\theta) = 0$ for $r > \alpha$. Substituting it into (4) and performing integration utilizing the asymptotic equation of the Bessel function, the well-known result.

$$F(z)=2J_1(z)/z \qquad (26)$$

is obtained where $z = k\alpha\rho$. It is multiplied by 2 so that $F(0) = 1$ at the origin. Applying the sampling theorems for the amplitude (8) and (9) in (26), we have

$$J_1(z)/z = 2J_0(z) \sum_{s=1}^{+\infty} 1/(\lambda_s^2 - z^2) \tag{27}$$

where λ_s is the s^{th} zero of $J_0(z)$. Generally, letting the pupil function be $r^n \exp(in\theta)$ in a circular aperture, we similarly obtain

$$J_{n+1}(z)/z = 2J_n(z) \sum_{s=1}^{+\infty} 1/(\lambda_{ns}^2 - z^2)$$

This calculation will be left for the reader. Figure 3 shows the calculation results with respect to four terms C_{01}, C_{02}, C_{03}, C_{04} taking the amplitudes at $z = \lambda_1$ as sampling values for the amplitude distribution given by the above (26). The amplitude at the origin is 0.992 where it should be 1. For a rigorous calculation

$$F(0) = 4 \sum_{s=1}^{+\infty} (1/\lambda_s^2) = 1 \tag{28}$$

In order to examine the degree of approximation at the origin, s must be found, for which $1/\lambda_s^2$ falls below the required accuracy using the asymptotic Equation (16) for the value of zero λ_s contained in (28). In order to insure a fourth place accuracy in the amplitude value at the origin, the s^{th} term for which $\lambda_s^2 \geq 10^4$ must be included in the calculation. According to (16), $s \approx 10^2/\pi$ — that is, sampling values up to approximately 30 terms are necessary. The interpolation for the amplitude coverges at approximately /108 s^{-2}. For a higher precision, therefore, many terms are needed. Hence, although the sampling value for the amplitude is the sample of the minimum necessary number, the convergence in the numerical calculation is not very good.

The situation is much better with the sampling theorem of 2.3. for the intensity distribution. As a result of the approximation calculation when

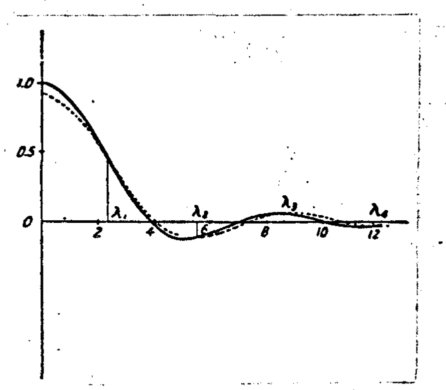

Fig. 3 The amplitude of the Airy figure at sharp focus is shown by full line and the amplitude calculated by sampling theorem from four samples at $\lambda_i, i=1,2,3,4$ by dotted line.

the sampling values are given at four points $z = \lambda_1/2, \lambda_2/2, \lambda_3/2, \lambda_4/2$, a 3-place accuracy ($10^{-3}$) is obtained within the minimum blur circle. The difference between the correct value is indiscernible. Let us compare with the numerical values.

Almost 1/1000 accuracy is obtainable. Let us examine the degree of approximation at the origin I(0), and the number of the required terms. The intensity at the origin is,

z	$(2J_1(z)/z)^2$	4 point approximation
0	1.0000	1.0006
1.0	0.7746	0.7756
2.0	0.3326	0.3330
3.0	0.0511	0.0510
4.0	0.0011	0.0018
5.0	0.0172	0.0185
6.0	0.0085	0.0114

$$I(0) = \sum_{s=1}^{+\infty} 2\{4J_1(\lambda_s/2)/\lambda_s\}^2/\lambda_s J_1(\lambda_s) \tag{29}$$

Applying the asymptotic equation for zeros as before, the magnitude of the s^{th} term is about $s^{-7/2}$. For 10^{-4} accuracy, approximately 14 terms are necessary. Since what we almost always need is the intensity distribution which we can obtain with good accuracy, the sampling theorem for the intensity is very useful. The high degree of accuracy obtainable for the same number of samplings is due to the redundancy in the sampling values as discussed in 2.3. The intensity at the origin obtained from the 4-point approximation for the amplitude discussed earlier becomes 0.8504. Compared to the intensity sampling which gives 1.0006, the accuracy is bad. According to the sampling theorem for the intensity, many samplings are necessary for a certain accuracy.

Let us obtain the response function for a circular aperture — that is, the Fourier transform of $\{2J_1(z)/z\}^2$ from (24). As it is well known [9], the rigorous equation for the response function for this case is given by (see Appendix 3),

$$D(r) = (2\vartheta - \sin 2\vartheta)/\pi \qquad \vartheta = \cos^{-1} r/2 \tag{30}$$

Figure 4 shows the result of approximation of the response function obtained with four intensity sampling values. According to the figure, although it shows good agreement around r = 2 with the rigorous solution, the error is large around r = 0. It is understandable, since, in general, the values of the response function in the vicinity of the origin are affected even by the distant values of the diffraction figure. Let us examine how much the approximation values improve by increasing the sampling values:

4 points	0.917
6 points	0.941
8 points	0.966

Let us examine the degree of approximation as before.

The value of the response function at the origin is,

$$D(0) = \sum_{s=1}^{+\infty} \{4J_1(\lambda_s/2)/\lambda_s\}^2 / \{2J_1(\lambda_s)\}^2 = 1 \tag{31}$$ [8]

Substituting the asymptotic value for λ_s as previously, the magnitude of the s^{th} term is approximately s^{-2}. This is same as the case of the sampling theorem for amplitudes, meaning that the convergence is slow at the origin. It is premature, however, to conclude from such a fact that it is impractical to obtain a response function from the sampling theorem. The value of the

(8) The value obtained from (24) is divided by 2 in order to have agreement with the response function of (30) which is normalized.

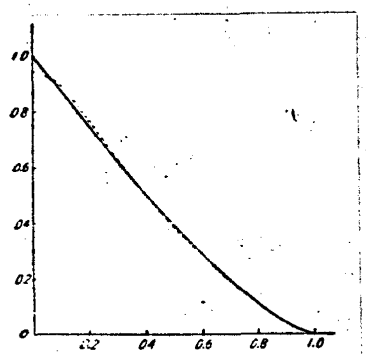

Fig. 4 The response function of a circular aperture without both aberration and focussing is shown by full line, and the response function calculated from four samples of intensity at $\lambda_{0i}/2$, $i=1,2,3,4$ by dotted line.

response function at the origin can be obtained comparatively easily by an analytical method. For example, one merely needs to consider the integral letting g = f* and $u_0, v_0 = 0$ in (20). Since D(0) is equal to the integral of the diffraction figure, by supplying the intensity distribution which is within our accuracy range, the approximation value of the response function from the sampling values will probably be satisfactory.

3.2 The General Case /109

Needless to say, the previous discussions pertained to an image by an ideal circular aperture when the object is a point light source. Let us give a preliminary discussion in applying the sampling theorem to an image formed by a system with aberrations and off-focusing for the case when the object is a point source, or when the object had an arbitrary amplitude distribution. In general, the pupil function [3] for the system with aberrations and off-focusing is

$$f(r,\theta)=\exp[ipr^2-iV(r,\theta)] \tag{32}$$

where p is a parameter which indicates off-focusing, and $V(r,\theta)$ is the wavefront aberration. In spherical aberrations, $V(r,\theta)$ is only a function of r, whereas in the non-spherical aberrations, such as astigmatism or coma, it is a function of r,θ. As clarified by Nijboer, $V(r,\theta)$ is normally expanded by circle polynomials [3,10], and each term of the series has a physical meaning.

For an object which has an arbitrary amplitude distribution, if $g(r,\theta)$ is the Fourier transform of the amplitude distribution, the Fourier spectrum of the waves at the exit pupil is equal to the product of the above $f(r,\theta)$ and $g(r,\theta)$. Each one has a frequency band determined by the aperture α. In order to apply the sampling theorems to these images, either the coefficients (2) of the Fourier-Bessel expansion of the given pupil function $f(r,\theta)$, or the sampling values of the image front must be determined. Generally, the analytical calculation of (2) is not easy. As (2) was studied long time ago by Lommel only for the off-focus case, sometimes it is solved using the Lommel functions U and V (Gray-Mathews, Chapter 14).

When the aberrations are not too large, (32) can be expanded by the circle polynomials as treated by Zernike, Nijboer and Nienhuis. Sampling values of 2 can be obtained either from the amplitude of the intensity of the diffraction waves of Nijboer, et al.

The calculation of Nijboer, et al. essentially is comprised of expanding $f(r,\theta)$ by the circle polynomials and performing its Fourier transform. Since the following general relation exists between the expansion of $f(r,\theta)$ by the circle polynomial and the Fourier-Bessel expansion, the pupil function can be expanded by the circle polynomials, which can then be subjected to the Fourier-Bessel expansion[(9)].

Expansion by the circle polynomials is easier than that by the Fourier-Bessel expansion. As the former is by a polynomial expansion, the final expansion equation can be obtained by elementary calculations by applying the asymptotic equation between the polynomial equations.

If the pupil function $f(r,\theta)$ is given inside the unit circle, generally

---

[(9)] Explanations are not given at this point in the original paper by Zernik, Nijboer, and Nienhuis. This probably is a new problem which arose in conjunction with the sampling theorem.

the pupil function is expanded by a circle polynomial $R^m_{m+2k}(r)$ as follows:

$$f(r,\theta)=\sum_{m=0}^{+\infty}\sum_{K=1}^{+\infty}(A_{m,m+2k}\cos m\theta + B_{m,m+2k}\sin m\theta)R^m_{m+2k}(r) \tag{33}$$

Where the expansion coefficient $A_{m,m+2k}$, $B_{m,m+2k}$ is derived from the orthogonal relation between $\{R^m_n\}$.

$$\left.\begin{matrix}A_{m,m+2k}\\ B_{m,m+2k}\end{matrix}\right\}=\frac{2(m+2k+1)}{\pi}\int_0^1\int_0^{2\pi} f(r,\theta) \times \begin{matrix}\sin m\theta\\ \cos m\theta\end{matrix} R^m_{m+2k}(r)\, r\,dr\,d\theta \tag{34}$$

The orthogonal relation,

$$\int_0^1 R^m_n R^{m'}_{n'}\, r\,dr = \delta_{nn'}\delta_{mm'}/(2n+2) \tag{35}$$

is applied. Actually, instead of integrating (34), the exponential function in $f(r,\theta)$ is expanded in series. In this case, by expressing $\exp(ipr^2)$ and $V(r,\theta)$ in circle polynomials like Zernike, et al. there will be many polynomial products. Using the following asymptotic equations, let us proceed so that $R^m_n(r)$ appears in the expanded terms which contain $\cos m\theta$ and $\sin m\theta$.

$$\left.\begin{aligned} rR^m_n(r) &= \frac{n-m+2}{2(n+1)}R^{m-1}_{n+1}(r) + \frac{n+m}{2(n+1)}R^{m-1}_{n-1}(r) \\ rR^m_n(r) &= \frac{n+m+2}{2(n+1)}R^{m+1}_{n+1}(r) + \frac{n-m}{2(n+1)}R^{m+1}_{n-1}(r) \end{aligned}\right\} \tag{36}$$

This only applies when $f(r,\theta)$ can be approximated by the first terms of the series. When the deviation is too large, it becomes complicated.

The Fourier transform of an expansion like (33) obtained by the above procedure gives the amplitude distribution of the image front. This is the result obtained by Zernike, Nijboer, and Nienhuis.

First, let us integrate with respect to θ the integral

$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi} R^m_{m+2k}(r) \frac{\sin}{\cos} m\theta \times \exp[ik\rho r\cos(\theta-\varphi)]\, r\,dr\,d\theta$$

which is required for the Fourier transform, and obtain $(i)^m \frac{\sin}{\cos} m\varphi J_m(k\rho r)$. The integration with respect to r is obtained by the following integral given by Zernike,

$$\int_0^1 R^m_{m+2k}(r)\, J_m(zr)\, r\,dr = (-1)^k \frac{J_{m+2k+1}(z)}{z} \tag{37}$$

The Fourier transform of (33) eventually becomes,

$$F(\rho,\varphi) = \sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty} (-1)^k (i)^m (A_{m,m+2k}\cos m\varphi + B_{m,m+2k}\sin m\varphi) J_{m+2k+1}(k\rho)/(k\rho) \tag{38}$$

Now let us apply the sampling theorem 2 to (38). Sampling values are obtained on the circle of radius $k\rho = \lambda_{ms}$ [zero of $(\lambda_{ms}:J_m(z))$] for the terms including $\sin m\phi$, $\cos m\phi$. /110

$$\left.\begin{aligned} F_m(\lambda_{ms}/k) &= (-1)^k (i)^m \frac{1}{2}(A_{m,m+2k} - iB_{m,m+2k}) \frac{J_{m+2k+1}(\lambda_{ms})}{\lambda_{ms}} \\ F_{-m}(\lambda_{ms}/k) &= (-1)^k (i)^m \frac{1}{2}(A_{m,m+2k} + iB_{m,m+2k}) \frac{J_{m+2k+1}(\lambda_{ms})}{\lambda_{ms}} \end{aligned}\right\} \tag{39}$$

where the positive integers s = 1, 2, 3, ... are taken.

The coefficients K_{ns} of the Fourier-Bessel expansion are obtained by applying (11) to (39). By restricting the sampling values with $\cos m\phi$, $\sin m\phi$ as

$$\left.\begin{aligned} A_m(\lambda_{ms}/k\alpha) &= \frac{1}{2\pi}\int_0^{2\pi} F(\lambda_{ns}/k\alpha,\varphi)\cos m\varphi\, d\varphi \\ B_m(\lambda_{ms}/k\alpha) &= \frac{1}{2\pi}\int_0^{2\pi} F(\lambda_{ns}/k\alpha,\varphi)\sin m\varphi\, d\varphi \end{aligned}\right\} \tag{40}$$

The sampling values are expressed by

$$A_m(\lambda_{ms}/ka) = (-1)^k (i)^m A_{m,m+2k} J_{m+2k+1}(\lambda_{ms})/\lambda_{ms}$$
$$B_m(\lambda_{ms}/ka) = (-1)^k (i)^m B_{m,m+2k} J_{m+2k+1}(\lambda_{ms})/\lambda_{ms}$$

Let us now consider the relation between the Fourier-Bessel expansion (1) and the circle polynomial expansion (33). The key lies in the integration (37). The basic function $\exp(in\theta)J_n(\lambda_{ns}r)$ of the Fourier-Bessel expansion is expanded by the circle polynomial. Otherwise, it is regarded as an equation which gives the expansion coefficients when the circle polynomial es expanded by the Fourier-Bessel expansion. Both expansions are related by

$$J_n(\lambda_{ns}r) = 2\sum_{k=0}^{+\infty}(-1)^k(n+2k+1) \times J_{n+2k+1}(\lambda_{ns}) R^{n}_{n+2k}(r)/\lambda_{ns} \tag{41}$$

$$R^{n}_{n+2K}(r) = 2\sum_{s=1}^{+\infty}(-1)^k J_{n+2k+1}(\lambda_{ns}) \times J_n(\lambda_{ns}r)/\lambda_{ns}\{J_n'(\lambda_{ns})\}^2 \tag{42}$$

Hence, Fourier-Bessel expansion is obtained by substituting (42) into (33). The reverse also holds. The expansion coefficient K_{ns} obtained in this way obviously agrees with that obtained from the sampling value (39).

Intensity distribution $I(\rho,\phi)$ is given by the product $F(\rho,\phi)$ from (38). As was repeatedly stated, it is more practical to apply sampling theorems for the intensity distribution. General equations for the intensity distribution are:

$$\begin{aligned}
I(\rho,\varphi) &= \sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty}\sum_{l=0}^{+\infty}(-1)^{k+l+n}(i)^{m+n} \\
&\quad \times (A_{m,m+2k}\cos m\varphi + B_{m,m+2k}\sin m\varphi) \\
&\quad \times (A^*_{n,n+2l}\cos n\varphi + B^*_{n,n+2l}\sin n\varphi) \\
&\quad \times J_{m+2k+1}(k\rho)J_{n+2l+1}(k\rho)/(k\rho)^2 \\
&= \sum_{m,n,k,l=0}^{\infty}\frac{1}{2}(-1)^{k+l+n}(i)^{m+n}\{(A_{m,m+2k}A^*_{n,n+2l} \\
&\quad + B_{m,m+2k}B^*_{n,n+2l})\cos(m-n)\varphi \\
&\quad + (B_{m,m+2k}A^*_{n,n+2l} - A_{m,m+2k}B^*_{n,n+2l}) \\
&\quad \times \sin(m-n)\varphi + (A_{m,m+2k}A^*_{n,n+2l} \\
&\quad - B_{m,m+2k}B^*_{n,n+2l})\cos(m+n)\varphi \\
&\quad + (B_{m,m+2k}A^*_{n,n+2l} + A_{m,m+2k}B^*_{n,n+2l}) \\
&\quad \times \sin(m+n)\varphi\}J_{m+2k+1}(k\rho)J_{n+2l+1}(k\rho)/(k\rho)^2
\end{aligned} \tag{43}$$

Hence, the sampling value for the intensity distribution is equal to the value of $J_{m+2k+1}(\lambda)J_{n+2\ +1}(\lambda)/\lambda^2$ of $J_{m-n}(z)$ and $J_{m+n}(z)$ times the corresponding coefficients of (43'). The quantity obtained by multiplying the coefficients from (11) to their sampling values is equal to the expansion coefficients when the Fourier transform of the intensity distribution is expressed by the Fourier transform. In such a manner,the response function of the pupil function with arbitrary aberrations is obtained numerically by the Fourier-Bessel expansion which has the sampling values of the intensity distribution as coefficients.

When (43)' is directly transformed, an analytic expression of the response function is obtained. By integration with respect to the angular variable, we obtain an expression which is replaced by $(i)^{m-n}$ cos $(m-n)\theta$ $J_{m-n}(\rho r)$, $(i)^{m+n}$ cos $(m+n)\theta$ $J_{m+z}(\rho r)$ and sin $(m-n)\theta$, sin $(m+n)\theta$. In infinite integration with respect to ρ, generally the following two types of integrals emerge:

$$I_{m-n}=\int_0^\infty J_{m-n}(\rho r)J_{m+2k+1}(\rho)J_{n+2l+1}(\rho)d\rho/\rho \tag{44}$$

$$I_{m+n}=\int_0^\infty J_{m+n}(\rho r)J_{m+2k+1}(\rho)J_{n+2l+1}(\rho)d\rho/\rho \tag{45}$$

These integrats can be evaluated by the method described in Appendix 3. Since a report pertaining to the analytical equations of such response functions cannot be found, they were included in the Appendix.

4. CONCLUSION

The application of the above sampling theorem for a circular aperture to experimental data requires a separate discussion. When a diffraction image which has a point light source as an object is experimentally given, and if the sampling value given by 2 is integrated on its sampling circle and determined experimentally, the parts which have n-fold symmetry with respect to the axis can be obtained separately. Also, response functions arising from such parts can be easily derived. This is an indication that sampling theorems are also useful for consolidating experimental data.

The other important application of the above sampling theorem is to extend the one-dimensional "intensity matrix" discussed in the preceding paper [6] to a two-dimensional image by a circular aperture. A report has been submitted pertaining to this problem. In this case, the element of the transformation matrix, which represents the transformation of the intensity matrix by passing through a circular aperture with aberrations, /111
can be obtained as an application of the sampling theorem for a circular aperture. It has the same function as the response function in discussing the formation of an image. The element of this transformation matrix is a quantity which indicates how a sampling function $C_{ns}(\rho,\phi)$ on the object appears on the sampling value of degree n and order s on the image.

The case where the object is a point light source was mainly discussed in 3. In applying the sampling theorem to an image of an object which has an arbitrary span, sampling corresponding to an extremely large degree n and order s is required. Since a rigorous treatment of all such sampling functions $C_{ns}(\rho,\phi)$ is formidable, asymptotic equations of the Bessel function of the form (15) and (16) are employed.

As evident from the discussion, several new problems on Bessel functions were created. A cross check on the problems by those who are interested will be appreciated.

I am grateful to Professors Hidetoshi Takahashi, Hiroshi Kubota, Goro Kuwabara and Iwao Okura of University of Tokyo for the valuable discussions.

REFERENCES

1. Gamo, Hideya. Kagaku, Vol. 26, No. 12, 1956.

2. Blanc-Lapierre, A. Ann. I'Inst. H. Poincare, Vol. 13, No. 4, 1953, p. 283.

   Gabor, D. Tech. Rep., No. 238, Res. Lab. Elect. M.I.T., April 3, 1952.

   di Francia, Toraldo. J. Opt. Soc. Amer., Vol. 45, 1955, p. 497.

   Fellgett, P. B. and E. H. Linfoot. Phil. Trans. Roy. Soc., London, Vol. A247, 1955, p. 369.

3. Zernike, F. and B. R. Nijboer. La Theorie des Images Optiques (Theory of Optical Images), pp. 227-235 (Revue d'Optique, 1949).

   Nijboer, B. R. Physica, Vol. 13, 1947, pp. 605-620.

   Nienhuis, K. and B. R. Nijboer. Physica, Vol. 14, 1948, pp. 590-608.

4. Hopkins, H. H. Proc. Roy. Soc., Vol. A 231, 1955, pp. 91-103.

   De, M. Proc. Roy. Soc., Vol. A 233, 1955, pp. 91-104.

5. Watson, G. N. Theory of Bessel Functions, Cambridge U. P. , 1922.

   Gray, Andrew and G. B. Mathews. A Treatise on Bessel Function and Their Applications to Physics, MacMillan Co, 2nd edition, 1922

   Magnus, W. and F. Oberhettinger. Formeln und Sätze für die spezieren Funktionen der Mathematischen Physik (Formulas and Theorems for Special Functions of Mathematical Physics), Springer, 1948.

6. Gamo, Hideya. J. Appl. Physics, Japan, Vol. 25, 1956, pp. 431-443.

7. Titchmarsh, E. C. Introduction to the Theory of Fourier Integrals, Oxofrd Clarendon Press, 1937.

8. Francon, M. Grundlagen der Optik, Hnadbuch der Physik, Vol. 24, Section 70, 1956, p. 276 f.

9. Maréchal, A. Grundlagen der Optik, Handbuch der Physik, Vol. 24, 1956, p. 153.

10. Zernike, F. Physica, Vol. 1, 1934, p. 700.

Zernike, F. and H. C. Brinkman. Proceedings, Koninglijke Nederlandsche Academie van Wettenschappen te Amsterdam, Vol. 38, 1935, pp. 161-170.

APPENDIX 1

Calculations on $C_{ns}(\rho,\phi)$

$$I=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{a}J_n(\lambda_{ns}r/a)\times\exp[in\theta+ik\rho r\cos(\theta-\varphi)]\,r\,dr\,d\theta$$

Integration with respect to angular variable θ is obtained by the formula

$$J_n(z)=\frac{1}{2\pi}\int_{\alpha}^{2\pi+\alpha}\exp[in\vartheta-iz\sin\vartheta]\,d\vartheta \tag{1.1}$$

from Watson (p. 20).

The integration,

$$I=\exp\left[in\left(\varphi+\frac{\pi}{2}\right)\right]\int_0^{a}J_n(\lambda_{ns}r/a)J_n(k\rho r)\,r\,dr \tag{1.2}$$

is obtained by the Lommel integral (Gray, Mathews, p. 69),

$$(\lambda^2-\mu^2)\int_0^{x}J_n(\lambda x)J_n(\mu x)\,x\,dx=x\{\mu J_n(\lambda x)J_n'(\mu x)-\lambda J_n(\mu x)J_n'(\lambda x)\} \tag{1.3}$$

where, λ_{ns} the zero of $J_n(x)$ is utilized.

Now, the orthogonal relation (7) of $C_{ns}(\rho,\phi)$ is zero when integrating with respect to φ when $n \neq m$. When $n = m$, by representing $C_{ns}(\rho,\phi)$ in the integral of (1.2), it returns to the integral

$$\int_0^{\infty}z\,dz\int_0^{1}y\,dy\,J_n(\lambda_{nt}y)J_n(zy)\int_0^{1}xJ_n(\lambda_{ns}x)J_n(xz)\,dx \tag{1.4}$$

Here, by the Fourier-Bessel integral (Gray Mathews, p. 96-97), we obtain

$$\int_0^{\infty}z\,dz\int_0^{1}J_n(\lambda_{ns}x)J_n(zx)J_n(zy)\,x\,dx=J_n(\lambda_{ns}y)\quad(0<y<1) \tag{1.5}$$

(1.4) is solved by the Lommel integral (1.3). However, when $\lambda = \mu$, we apply

$$\int_0^x \{J_n(\lambda x)\}^2 x\,dx = \frac{1}{2}x^2\Big[\{J_n'(\lambda x)\}^2 + \Big(1-\frac{n^2}{\lambda^2 x^2}\Big)\{J_n(\lambda x)\}^2\Big] \tag{1.6}$$

where the orthogonal relation is proved as a special case described in Appendix 2.

APPENDIX 2

Fourier Transform of $C_{ns}(\rho,\phi)C_{mt*}(\rho,\phi)$

Let $D_{nm}(r,\theta;\lambda_{ns\theta}\lambda_{mt})$ be the Fourier transform of $C_{ns}(\rho,\phi)C_{mt*}(\rho,\phi)$,

$$\begin{aligned} D_{nm}(r,\theta;\lambda_{ns},\lambda_{mt}) &= A\exp[i(n-m)\theta] \\ &\times \int_0^\infty \frac{J_n(x)J_m(x)J_{n-m}(xr/\alpha)}{(x^2-\lambda_{ns}^2)(x^2-\lambda_{mt}^2)}x\,dx \end{aligned} \tag{2.1}$$

/112

where the constant A is

$$A = 4\lambda_{ns}\lambda_{mt}/J_n'(\lambda_{ns})J_m'(\lambda_{mt})\,k^2\alpha^2$$

By separating into partial fractions, the integral of (2.1) can be represented as the difference of two integrals.

$$\begin{aligned} D_{nm}(r,\theta;\lambda_{ns},\lambda_{mt}) &= A\exp[i(n-m)\theta] \\ &\times \{I_{n,m}(r/\alpha;\lambda_{ns}) - I_{n,m}(r/\alpha;\lambda_{mt})\}/(\lambda_{ns}^2-\lambda_{mt}^2) \end{aligned} \tag{2.2}$$

where

$$I_{n,m}(r/\alpha;\lambda) = \int_0^\infty J_n(x)J_m(x)J_{n-m}(xr/\alpha)\frac{x\,dx}{x^2-\lambda^2} \tag{2.3}$$

When n = m and s = t, we have

$$D_{nn}(r,\theta;\lambda_{ns},\lambda_{ns}) = A\tilde{I}_n(r,\theta) \tag{2.4}$$

$$\tilde{I}_n(r,\theta) = \int_0^\infty \{J_n(x)\}^2 J_0(xr/\alpha)\frac{x\,dx}{(x^2-\lambda^2)^2} \tag{2.5}$$

Thus, it is necessary to determine the two infinite integrals, (2.3) and (2.5). Direct solutions to these are not listed in reference books. Since the integrals have a definite physical significance, as expected the solution was found by the following procedure. By expressing the product of Bessel functions

$J_n(x)J_m(x)$ by an integral which includes J_{n-m}, and changing the order of integration it can be expressed as an infinite integral of the product of two Bessel functions of the same order. Since the latter can be determined by the Hankel integral, the solution can be derived.

According to Watson, § 5.43 (page 150),

$$J_n(x)J_m(x)=\frac{2}{\pi}(-1)^m\int_0^{\frac{\pi}{2}} J_{n-m}(2x\cos\theta)\times\cos(n+m)\theta d\theta \tag{2.6}$$

Let us substitute this into (2.3), and change the order of integration.

The Hankel integral [Watson § 13.53 (p. 429)] can be solved as a Hankel function $Hn^{(1)}$ of the 1st kind

$$\int_0^\infty J_n(ax)J_n(bx)\frac{xdx}{x^2-\lambda^2} = \begin{cases} \frac{1}{2}\pi i\ J_n(b\lambda)\ H_n^{(1)}(a\lambda) & (a>b) \\ \frac{1}{2}\pi i\ J_n(a\lambda)\ H_n^{(1)}(b\lambda) & (a<b) \end{cases} \tag{2.7}$$

By this integration, the following result is obtained;

$$\begin{aligned} &I_{nm}(r/\alpha;\lambda) \\ &=(-1)^n iJ_{n-m}(\lambda r/a)\int_0^{\cos^{-1}r/2a} H_{n-m}^{(1)}(2\lambda\cos\theta)\cos(n+m)\theta d\theta \\ &+(-1)^n iH_{n-m}^{(1)}(\lambda r/a)\int_{\cos^{-1}r/2a}^{\pi/2} J_{n-m}(2\lambda\cos\theta)\cos(n+m)\theta d\theta \end{aligned} \tag{2.8}$$

The integrals contained in (2.8) can be determined as a series by Garf's generalized equation (Watson, p. 361) of the addition theorem of Neumann's formula

$$\left.\begin{aligned} &J_\nu(2\lambda\cos\theta)\cos\nu\theta \\ &\quad=\sum_{k=-\infty}^{+\infty}(-1)^k J_{\nu+k}(\lambda)J_k(\lambda)\cos 2k\theta \\ &H^{(1)}{}_\nu(\lambda\cos\theta)\cos\nu\theta \\ &\quad=\sum_{k=-\infty}^{+\infty}(-1)^k H^{(1)}{}_{\nu+k}(\lambda)J_k(\lambda)\cos 2k\theta \end{aligned}\right\} \tag{2.9}$$

By substituting this result into (2.8), we have

$$
\begin{aligned}
&\int_0^{\cos^{-1} r/2\alpha} H_{n-m}^{(1)}(2\lambda\cos\theta)\cos(n+m)\theta d\theta \\
&\quad = \sum_{k=-\infty}^{+\infty} (-1)^k H_{n-m+k}^{(1)}(\lambda) J_k(\lambda) \\
&\quad \times \int_0^{\cos^{-1} r/2\alpha} \begin{matrix} \cos(n+m)\theta \\ \cos(n-m)\theta \end{matrix} \cos 2k\theta d\theta
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
&\int_{\cos^{-1} r/2\alpha}^{\pi/2} J_{n-m}(2\lambda\cos\theta)\cos(n+m)\theta d\theta \\
&\quad = \sum_{k=-\infty}^{+\infty} (-1)^k J_{n-m+k}(\lambda) J_k(\lambda) \\
&\quad \times \int_{\cos^{-1} r/2\alpha}^{\pi/2} \begin{matrix} \mathrm{dos}(n+m)\theta \\ \cos(n-m)\theta \end{matrix} \cos 2k\theta d\theta
\end{aligned} \tag{2.11}
$$

The integrals of the trigonometric functions contained in (2.10) and (2.11) can be solved by an elementary method. Hence, by substituting (2.10) and (2.11) into (2.8), $I_{nm}(r/\alpha;\lambda)$ is determined.

The subsequent integral (2.5) $I_n(r,\theta)$ can be determined by differentiation of $I_{nm}(r/\alpha;\lambda)$, that is

$$
\tilde{I}_n(r/\alpha;\lambda) = \frac{1}{2\lambda} \frac{d}{d\lambda}[I_{nn}(r/\alpha;\lambda)] \tag{2.12}
$$

From the above result, it can be proved that $D_{nm}(r/\alpha,\theta;\lambda_{ns} \; \lambda_{mt})$ is restricted only in the region of radius $I_{nm}(r/\alpha;\lambda)$. For example, determination of the above $\tilde{I}_{nm}(2\alpha;\lambda)$ gives

$$
\begin{aligned}
I_{nm}(2\alpha;\lambda) &= (-1)^n i H_{n-m}^{(1)}(2\lambda) \\
&\quad \times \int_0^{\pi/2} J_{n-m}(2\lambda\cos\theta)\cos(n+m)\theta d\theta
\end{aligned}
$$

From Equation (2.6), the integral on the right hand returns to the product $I_{nm}(r/\alpha;\lambda)$. Since λ is the zero of $J_n(x)$ or $J_m(x)$, $I_{nm}(2\alpha;\lambda) \equiv 0$. Now, the integral does not exist for $r > 2\alpha$. The same is true for $I_{nm}(2\alpha,\lambda)$

The orthogonal relation of $C_{ns}(\rho,\phi)$ proved in Appendix 1 corresponds to the integral (2.2) of Appendix 2 when $r = 0$. Therefore, the above solution gives a separate proof for the orthogonal relation (7). It is zero except when $n = m$ and $s = t$. When $n = m$ and $s = t$,

$$I=\int_0^\infty \{J_n(x)\}^2 \frac{x\,dx}{(x^2-\lambda^2)^2}=\frac{1}{2\lambda^2} \tag{2.13}$$

This agrees with the result proved by using the Fourier-Bessel integral in Appendix 1.

If we first determine $I_{nm}(0,\lambda)$ with (2.12), then by (2.8) the integral /113
of (2.13) becomes

$$I_{nn}(0;\lambda)=(-1)^n i\int_0^{\pi/2} H_0^{(1)}(2\lambda\cos\theta)\cos 2n\theta\, d\theta$$

Nevertheless, the coefficients of the series (2.10) are all zero except in the case when $k = \pm n$.

Hence,

$$I_{nn}(0,\lambda)=\frac{\pi}{2}\, i\, J_n(\lambda) H_n^{(1)}(\lambda)$$

where the integral (2.13) is determined by

$$I=\frac{1}{2\lambda}\,\frac{d}{d\lambda}\,[I_{nn}(0;\lambda)]$$

Applying $H_n^{(1)}(\lambda)=J_n(\lambda)+iY_n(\lambda)$ and using

$$J_n(\lambda)\,Y_{n+1}(\lambda)-J_{n+1}(\lambda)\,Y_n(\lambda)=-2/(\pi\lambda)$$

(Watson § 3.63, page 77)

the first result $1/(2\lambda^2)$ is obtained.

APPENDIX 3

Analytical Equation of Response Function

The response function of a circular aperture with no aberration and off-focusing (Airy figure),

$$D(r)=\int_0^\infty \left\{\frac{2J_1(x)}{x}\right\}^2 J_0(rx)x\,dx \tag{3.1}$$

With Gegenbauer formula (Watson, p. 367) let us replace $\{2J_1(x)/x\}^2$, that is

$$\left\{\frac{2J_1(x)}{x}\right\}^2=\frac{2}{\pi}\int_0^\pi \frac{J_1(2x\cos\phi/2)}{x}\,\frac{\sin^2\phi}{\sin\phi/2}\,d\phi \tag{3.2}$$

Substituting this into (3.1) and changing the order of integration, we have

$$D(r)=\frac{2}{\pi}\int_0^\pi \frac{\sin^2\phi}{\sin\phi/2}\int_0^\infty J_1(2x\sin\phi/2)J_0(rx)\,dx \tag{3.3}$$

The in inite integral contained here can be solved as follows. (Watson, p. 406, Magnus-Oberhettinger, p. 50);

$$\int_0^\infty J_1(bx)J_0(ax)\,dx=\left\{\begin{array}{ll} 0 & (a>b) \\ \dfrac{1}{2b} & (a=b) \\ \dfrac{1}{b} & (a<b) \end{array}\right\} \tag{3.4}$$

Let us substitute this into (3.3)

$$D(r)=\frac{2}{\pi}\int_{2\sin^{-1}r/2}^{\pi}(1+\cos\phi)\,d\phi=\frac{2}{\pi}(2\vartheta-\sin 2\vartheta) \qquad \vartheta=\cos^{-1}r/2 \tag{3.5}$$ [10]

Let us now consider the response function for the case with spherical aberrations and off-focusing. In this case it is relatively simple, since

---

[10] The difference of 2 in the coefficient from the text (30) arises from the normalization, D(0) = 1. Normally, this integral is obtained from the convolution integral [9]. It can also be solved by (3.12).

it is axially symmetric and there is no term with cos and sin in (43'), that is

$$D(r)=\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}(-1)^{k+l}A_{2k}A_{2l}^{*}\times\int_0^{\infty}J_{2k+1}(x)J_{2l+1}(x)J_0(rx)\,dx/x \tag{3.6}$$

For the integral, we have

$$I_{kl}=\int_0^{\infty}J_{2k+1}(x)J_{2l+1}(x)J_0(rx)\,dx/x$$

When k = 1 using Gegenbauer's formula [Watson, p. 367 (17)] with respect to $\{J_{2k+1}(x)/x\}^2$,

$$\left\{\frac{J_{2k+1}(x)}{x}\right\}^2=\frac{1}{(2k+1)\pi}\int_0^{\pi}\frac{J_1(2x\sin\phi/2)}{2x\sin\phi/2}\times C^1_{2k}(\cos\phi)\sin^2\phi\,d\phi \tag{3.8}$$

Substituting this into (3.7) and using (3.4)

$$I_{kk}(r)=\frac{1}{(2k+1)\pi}\int_{2\sin^{-1}r/2}^{\pi}C^1_{2k}(\cos\phi)\frac{\sin^2\phi}{4\sin^2\phi/2} \tag{3.9}$$

where Gegenbauer's polynomial $C_{2k}{}^1$ can be expressed by a sin function as follows

$$C^1_{2k}(\cos\phi)=\sin(2k+1)\phi/\sin\phi \tag{3.10}$$

By substituting this into (3.9), $I_{kk}(r)$ can be obtained by an elementary method.

When k = 1, the product $J_{2k+1}(x)J_{2l+1}(x)$ is by (2.6) or by the following integral

$$J_n(x)J_m(x)=\frac{2}{\pi}\int_0^{\pi/2}J_{n+m}(2x\cos\theta)\cos(n-m)\theta\,d\theta \tag{2.6'}$$

From this,

$$I_{kl}(r)=\frac{2}{\pi}\int_0^{\pi/2}\cos 2(k-l)\theta\,d\theta\int_0^{\infty}J_{2(k+l)}(2x\cos\theta)\times J_0(rx)\,dx/x \tag{3.11}$$

or

$$=(-1)\frac{2}{\pi}\int_0^{\pi/2}\cos 2(k+l+1)\theta\,d\theta\int_0^{\infty}J_{2(k-l)}(2x\cos\theta)\times J_0(rx)\,dx/x \tag{3.11'}$$

Expressing $k+l$ or $k-l$ by n, the necessary integral is

$$A=\int_0^{\infty}J_0(rx)J_{2n}(2x\cos\theta)\,dx/x$$

When r and 2 cos θ are real numbers and 2n > 0, the integral can be solved as a special case by Sonine, Schofheitlin's formula (Magnus-Oberhettinger, p. 49).

$$\left.\begin{array}{ll}\text{(i)}\ r<2\cos\theta & \\ \quad A=\frac{1}{2n}\,{}_2F_1(n,-n;1;Z^2) & \\ \text{(ii)}\ r\geqq 2\cos\theta & Z=r/2\cos\theta \\ \quad A=0 & \end{array}\right\} \tag{3.12}$$

where ${}_2F_1(\mathrm{nm}\ {-n};1;(Z)^2)$ is a polynomial up to $(Z)^{2n}$ term as follows

$$\begin{aligned}&{}_2F_1(n,-n;1;(Z)^2)\\&=1-n^2Z^2+\frac{n^2(n^2-1)}{(2!)^2}Z^4-\frac{n^2(n^2-1)(n^2-2^2)}{(3!)^2}Z^6+\\&\cdots\cdots+(-1)^n\frac{n^2(n^2-1)\cdots\{n^2-(n-1)^2\}}{(n!)^2}Z^{2n}\\&Z=r/2\cos\theta\end{aligned} \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11) or (3.11'), we have

$$I_{kl}(r)=\frac{2}{\pi}\int_0^{\cos^{-1}r/2}\cos 2(k-l)\theta\times\frac{{}_2F_1(k+l,-(k+l);1;(r/2\cos\theta)^2)}{2(k+l)}\,d\theta \tag{3.14}$$ /114

or it becomes the same form with respect to $k-l$.

This integral can be solved as a polynomical in r^2, and the coefficients can be solved simply as a trigonometric integral. As evident from the above, when $k = l\,(>0)$ by changing it into an integral containing $J_{2k}(2x\cos\theta)$ using

(2.6'), (3.12) can be used, and can be solved with the Sonine-Schafheithlin formula without resorting to Gegenbauer's formula.

For the case of non-spherical aberration, it can be solved similarly using (2.6') and Sonine-Schafheitlin's formula.

Thus, we have three ways to determine the response function:

(i) By the sampling theorem using the Fourier-Bessel function (12).

(ii) From the sampling coefficients for amplitude, utilizing the Fourier transform of $C_{ns}C_{mt*}$ of Appendix 2.

(iii) By the analytical method from the intensity distribution of Appendix 3.

For the numerical calculations, method (i) is most practical. Given a sampling value, the answer can be obtained from the known table of Bessel functions. The disadvantage is, as stated in 31, the correct value in the vicinity of $r = 0$ cannot be obtained by this method, whereas with method (iii) it is easy to obtain values at $r = 0$. Hence, methods (i) and (iii) complement each other. Regarding method (ii), when the intensity distribution is given as an intensity matrix, the Fourier transform of the intensity distribution can be obtained by multiplying the series solution of Appendix 2 to the matrix elements. In this case, also, it is probably convenient to use (i) to obtain the solution.

Translated for National Aeronautics and Space Administration under contract No. NASw 2035, by SCITRAN, P. O. Box 5456, Santa Barbara, California, 93108.